Metric Spaces and Topology
Lecture 12

Perfect set property. A metric space (X, d) has the *perfect set property* (PSP) if it's either ctbl or $\underbrace{\text{embeds } 2^{\mathbb{N}}}_{\Updownarrow \text{ contains a (homeomorphic) copy of } 2^{\mathbb{N}}}$, i.e. $\exists\, 2^{\mathbb{N}} \hookrightarrow X$ continuous embedding.

Cantor defined this and successfully showed that all closed subsets of $\mathbb{R}$ have PSP. He wanted to prove this for all subsets of $\mathbb{R}$, thus proving the continuum hypothesis. However, one can ~~prove~~ build, using AC, a subset of $\mathbb{R}$ that doesn't satisfy PSP, so this approach to proving the continuum hypothesis will not work. Subsets without PSP but of continuum cardinality are called *Bernstein sets* and we will build one in $\boxed{\text{HW}}$.

Def. A metric space is called *perfect* if it has no isolated points.

Examples. $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{R}^d$, $\mathbb{R}^{\mathbb{N}}$, any interval in $\mathbb{R}$, e.g. $(0,1]$, $(0,1)$, $[0,1]$, $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$, $\mathbb{R} \setminus \mathbb{Q}$.

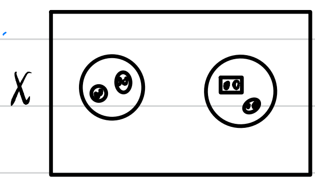
Nonexamples. $\underbrace{\{0\} \cup \{\tfrac{1}{n} : n \in \mathbb{N}^+\}}_{\text{closed nonperfect}}$, $\mathbb{Z}$, $\mathbb{N}$, $\{0\}$.

Observations. (a) Any open subset of a perfect space is perfect.
(b) Any dense subspace of a perfect space is a perfect space.

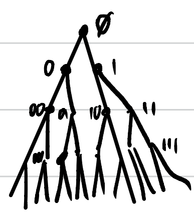
Warning. The term *perfect set* is also used, but it means a perfect CLOSED subset of a metric space.

Perfect set theorem. Any nonempty perfect complete metric space embeds $2^{\mathbb{N}}$; in particular, is $\geq$ continuum.

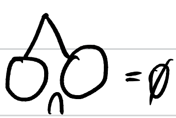
Proof.



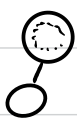
Let (X,d) be a $\emptyset \neq$ perfect complete m.s. We will build a *Cantor scheme*, i.e. a sequence $(U_s)_{s \in 2^{<\mathbb{N}}}$ of subsets of X such that



(i) $U_{s0} \cap U_{s1} = \emptyset$



(ii) $U_{si} \subseteq U_s$



Moreover, we will ensure that

(iii) U_s is open nonempty.

(iv) $U_{si} \subseteq_c U_s$, i.e. $\overline{U_{si}} \subseteq U_s$

(v) the scheme has *vanishing diameter*, i.e. $\lim_{|s| \to \infty} \operatorname{diam}(U_s) = 0$.

For example, $\operatorname{diam}(U_s) \leq 2^{-|s|}$, where $|s| :=$ length of s.
Let's first build such a scheme and then see how to define the embedding $2^{\mathbb{N}} \hookrightarrow X$.

We build $(U_s)_{s \in 2^{<\mathbb{N}}}$ by recursion on $|s|$. Let $U_\emptyset := X$.
Suppose U_s is defined and we define U_{s0} and U_{s1} as follows.

Because $U_s \neq \emptyset$ and open, it must contain at least two points x, y because X is perfect. Thus there are two nonempty disjoint open balls U_{s0} and U_{s1} whose closures are still contained in U_s. We can take these balls small enough so their diam $\leq 2^{-(|s|+1)}$.
This finishes the construction of the desired scheme $(U_s)_{s \in 2^{<\mathbb{N}}}$.

We now define the associated map $f: 2^{\mathbb{N}} \to X$ as follows: $\underset{2^{\mathbb{N}}}{\overset{\uparrow}{x}} \mapsto$ the unique element in $\bigcap_{n \in \mathbb{N}} U_{x|_n}$.

$\bigcap_{n \in \mathbb{N}} U_{x|_n} \underset{(iv)}{=} \bigcap_{n \in \mathbb{N}} \overline{U_{x|_n}} \underset{(v)}{\neq} \emptyset$, the last statement is because X is complete.

<u>f is injective:</u> By property (ii): if $x, y \in 2^{\mathbb{N}}$ are distinct, then $\exists n$ s.t. $x(n) \neq y(n)$, so $x|_{n+1} \neq y|_{n+1}$, hence

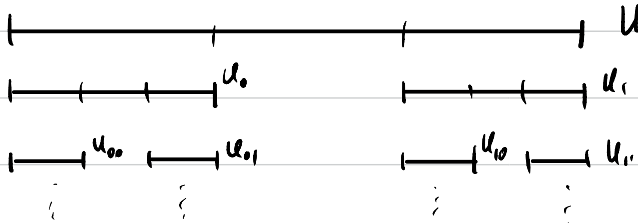
$$U_{x|_{n+1}} \cap U_{y|_{n+1}} = \emptyset.$$

f is continuous. Fix $x \in 2^{\mathbb{N}}$ and an ε-ball $B_\varepsilon \ni f(x)$. We know that $U_{x|_n}$ are open neighborhood of $f(x)$ of vanishing diameter. Thus, $\forall^\infty n$ $\operatorname{diam}(U_{x|_n}) < \varepsilon$ hence $U_{x|_n} \subseteq B_\varepsilon$, thus $[x|_n] \subseteq f^{-1}(B_\varepsilon)$ for some n.

f^{-1} is continuous. This is automatic because $2^{\mathbb{N}}$ is compact and X is Hausdorff, but we will prove it anyway. HW

$\boxtimes$

Remark. The construction of the usual Cantor set $\mathcal{C} \subseteq [0,1]$ is done exactly via the same Cantor scheme as in the proof:



Cor. Every nonempty open subset of a perfect complete metric space is unctbl; in fact, contains a copy of $2^{\mathbb{N}}$.

Proof (H. Karapetyan). If $U \subseteq X$ is open and nonempty, then

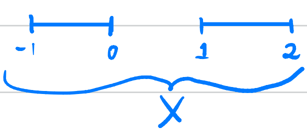
U contains an open ball $B_r(x)$ of radius r, hence also the closure of the open ball $B_{r/2}(x)$, which can easily be seen to be perfect.

$\boxed{\text{HW:}}$ Show that in a perfect metric space closure of open is still perfect.

Thus, U contains a nonempty perfect complete metric space $\overline{B_{r/2}(x)}$, hence also a copy of $2^{\mathbb{N}}$. $\square$

Proof 2. Recall the optional homework problem that $\exists$ a complete metric on U equiv. to the restriction. $\square$

Warning. Closure of a an open ball is in general smaller than the closed ball of the same radius. Closure of an open ball is always perfect in a perfect space, while a closed ball may not be. For example, $X := [-1,0] \cup [1,2]$ is a perfect complete metric space but



$$B_1(0) = (-1,0], \text{ so } \overline{B_1(0)} = [-1,0],$$

while $\overline{B}_0(1) = [-1,0] \cup \{1\}$, which isn't perfect.

Def. A metric space is called Polish if it is complete and separable ($\Leftrightarrow$ 2nd ctbl).

Cantor–Bendixson theorem. Polish spaces have the PSP. In particular, Polish spaces satisfy the continuum hypothesis.